An Analytical Approach to Star Identification Reliability

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ABSTRACT

The problem of real-time, on-board star pattern identification is considered which must precede any spacecraft attitude estimation algorithm based on measured line of sight directions to stars. Following the use of the search-less k-vector method[1, 2] to access feasible candidate stars, a tiered logical structure is introduced in which inter-star angles for pairs, triples and general elementary polygons are used to match measured star patterns to corresponding patterns in a star catalog. The fundamental innovation of this paper is that analytical expressions are developed for the expected frequency of matching an observed star pattern with an incorrect pattern in the star catalog due to measurement error. Such knowledge of erroneous match frequencies can be used to rigorously terminate the star identification process with a virtually certain match, while obviating the need for expensive Monte-Carlo simulations. Developments shown are supported with a few simulation results.

INTRODUCTION

This paper provides the mathematical tools which establish in a closed form, the expected frequency that a given measured star pattern may match with an invalid star catalog pattern simply due to measurement error. We consider the case of no prior information, so all conceivable star patterns imaged by the sensor from the whole sky must be considered as candidates. The formulation focuses upon the three most fundamental geometric structures of star patterns used in star identification algorithms.

A star polygon geometric structure is defined by the set of $M = \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$ inter-star angles associated with a *spherical polygon* set of n stars, such as pairs ($n = 2$), triangles ($n = 3$), as well as pyramids ($n = 4$), or more stars ($n > 4$). The spherical polygon is closely related to the usual polygon where the straight line sides are replaced by great circle arcs (angles) on the surface of a unit sphere connecting the neighboring pairs of stars in a set of n stars. More specifically, the star pattern geometric structure for the purpose of star identification is defined by the set of M inter-star angles $\vartheta_{ij} = \vartheta_{ji} = \cos^{-1}(b_i^{\mathrm{T}} b_j)$ measured between each distinct pair of the n line of sight vectors $\{(b_i, b_j) : (i, j) \in \{1, 2, \ldots, n\}\}$ which point from the sensor toward the vertices of the star spherical polygon on the celestial sphere. Note we adopt the convention that the measured line of sight unit vectors with components in the sensor *body* axes are denoted b_i, whereas the corresponding line of sight vectors based on cataloged information with components in the inertial $reference$ frame of the star catalog are denoted r_I. Indeed, the objective of star identification can be reduced to finding the correspondence between indices (i) of measured stars and the indices (I) of cataloged stars. We note that reflections may cause 180° ambiguities to arise which may frustrate the final determination of the cataloged indices corresponding to measured stars. In the current paper, we also discuss means for correctly resolving such ambiguities. (except for the case of a single measured star pair, which is anyway never sufficient for high confidence star pattern identification)

Matching the set of M measured inter-star angles $[\cos^{-1}(b_i^{\mathrm{T}} b_j)]$ with a cataloged set of inter-star angles $[\cos^{-1}(r_i^{\mathrm{T}} r_j)]$ to within measurement precision provides the basis for a hypothesis that the stars at the vertices of the measured polygon of stars are indeed the cataloged stars at the corresponding vertices of the matching polygon from the star catalog. Note however that such a match does not ensure an individual star identification and this hypothesis should be accepted with the calculated probability that such a match may occur at random between the measured polygon and an invalid set of n cataloged stars. For example, if we know the theoretical probability of matching an invalid polygon with a given set of measured inter-star angles to within known measurement error statistics is on the order of 10^{-20}, then we can be justifiably optimistic that the star identification hypothesis is valid. On the other hand, if the random invalid match probability is greater than some tolerance (say 10^{-5}), then we should have cause for concern and should likely reject the hypothesis and/or match more stars until the prescribed tolerance for a random invalid match probability is met.

In order to implement such a statistical decision process based upon theoretical probabilities, we need to know the invalid match frequency formulas developed in this paper. The frequency formulas are derived for the first few elementary star polygon cases, after which the formulas for the more general polygons are simply stated. These explicit analytical formulas provide a basis for statistical inference logical tests of a star pattern identification hypothesis than has not been available before, although heuristic practical algorithms have been developed empirically based on Monte Carlo testing and trial and error tuning of available star identification algorithms. A fundamental difficulty associated with such Monte Carlo testing is that random sampling statistical inference computational approaches become infeasible if one is pursuing frequencies on the order of 10^{-7} or smaller, because of the necessity of doing a very large number (certainly, $> 10^8$) of samples to establish statistically significant frequency estimates for such infrequently occurring events. Indeed it would be desirable if computationally feasible, to reduce the expected frequency of a random star identification to be significantly less than unity over the lifetime of a given mission [i.e., 1/(# of star identifications over the lifetime of a multi-year mission), which for the anticipated high frame rate active pixel cameras corresponds to a frequency less than $< 10^{-10}$]. Of course, we must acknowledge that there is a large distinction between the *lost in space* case and the case of *recursive* star identification where we can use new stars in polygons with previously identified stars whose catalog identity is near certain, so the *no prior information perspective* implicit in the above discussion may be highly conservative.

In any event, we anticipate that over the course of the next decade, there will be at least an occasional need for star identification algorithms with an overall expected frequency of mismatches approaching 10^{-10}. For longer missions with high star camera frame rates, we can conceive the need for an exceptionally small expected star mismatch frequencies (perhaps even $< 10^{-20}$). Obviously validating such frequency estimates is compounded because with every sensor design change, and every re-setting of any variable system parameter may necessitate repeating the miss-match frequency analysis. It is evident that Monte Carlo processes will be impractical in this situation. Like the perpetual thirst for faster computers, we can never develop a star identification algorithm that fails too rarely or be too confident in our star identification process! Moreover, even without pursuing such small frequencies of spurious star identifications, it is obvious that having the capability to quantify and minimize the frequency of failure is fundamental to analyzing/optimizing overall mission reliability. Therefore, we expect that the formulas developed herein will find a very practical home in sensor design and mission analysis by eliminating the reliance on slowly converging statistical simulations such as Monte Carlo processes.

A high percentage of spurious images (spikes) introduces a crisis in almost all existing algorithms for star pattern recognition for stars imaged by CCD star trackers. Failures and anomalies associated with such spurious images have been experienced several times in space missions which use star trackers to estimate the spacecraft attitude. For example few years ago, the STS 101 SOAR star tracker experiment encountered spurious sun reflections off an adjacent experiment's debris, resulting in a large number of spikes that, in turn, caused the star pattern recognition algorithms (used for SOAR) to fail. Using the above estimation of false frequency, it is possible to develop a new star identification algorithm that, better than any known approach to solve the problem, presents the simultaneous advantages of being extremely efficient and robust to random spurious images, and in addition, enjoys the advantage that *we can compute an estimate that informs the decision to accept or reject a star pattern match hypothesis*. If the estimated frequency of a random pattern match is not sufficiently small, remedial hypothesis/test logic can be invoked to add more stars to the pattern or reject the entire image*. In fact, the resulting adaptive capability to identify and delete spikes (due to electronic noise, planets, light reflections, etc.), is such that the method has been demonstrated to reliably accomplish the star identification process under extreme scenarios, with too many spurious images, a most rare occurrence in practice.

EXPECTED MATCH FREQUENCY OF MEASURED STAR PATTERN TO INVALID STAR POLYGONS IN CATALOG

Let us consider the whole sky with a uniform star distribution. This implies that the star density ρ (which depends on the given magnitude threshold m) is simply given as

$$\rho(m) = \frac{N(m)}{4\pi} \tag{1}$$

where $N(m)$ is the overall number of stars with magnitude less than m. The relationship between the magnitude threshold m and $\ln\{N(m)\}$ can be approximated[3] by the following linear expression

$$m \cong 0.8985 \ln N - 2.0474 \pm 0.2 \tag{2}$$

The above is a least-squares straight line fit. A quadratic least-squares fit provides a better

*This adaptive approach is different from Pyramid algorithm[4] approach. Pyramid decision of a successful star identification is left to the identification of a four-star structure.

approximation as:

$$m \cong 0.0126\,(\ln N)^2 + 0.7109\,\ln N - 1.3734 \pm 0.1 \tag{3}$$

Figure 1 shows residuals between the linear and quadratic best fits together with the associated standard deviations. We conclude that Eq. (2) provides an adequate approximation, especially for $N > 1000$, because the camera determined magnitude of a star seldom matches to better than 0.1 when compared to a catalog value. †

Next, consider the spherical area cut out by a cone (having its vertex at the center of the sphere) of aperture ϑ, i.e. the spherical area given by

$$S(\vartheta) = 2\pi\,(1 - \cos\vartheta) \tag{4}$$

We consider the scenario in which the axis of the above mentioned cone is aligned with the i-th star, as shown in Figure 2. The infinitesimal spherical area $dS(\vartheta)$, that can be evaluated as the difference between two cones of apertures $(\vartheta + d\vartheta)$ and ϑ, has the area

$$dS(\vartheta) = S(\vartheta + d\vartheta) - S(\vartheta) = 2\pi\,\sin\vartheta\,d\vartheta \tag{5}$$

Then, the expected number of stars falling in $dS(\vartheta)$ is given by:

$$dn(\vartheta) = \rho^*\,dS(\vartheta) = \frac{(N-1)}{2}\,\sin\vartheta\,d\vartheta \tag{6}$$

where, $\rho^* = \frac{N-1}{4\pi}$ indicates a uniform star density which slightly differs from the expression of the uniform star density ρ given in Eq. (1). This difference is due to the fact that the

†Note that the frequency appearing in these residuals depend on the fact that the star catalog provides the magnitude information with precision truncated to 0.1.

$i-$th "pivot" star lying on the axis of the cone is excluded from the count for evaluating the new star-density.

Now, if we consider all N catalog stars as candidate pivot stars, we are led to the differential frequency

$$df_{ij}(\vartheta) = \frac{1}{2}\, N\, dn(\vartheta) = \frac{N(N-1)}{4}\, \sin\vartheta\, d\vartheta \tag{7}$$

Note that the above equation provides the expected number of star pairs "i-j" separated by an angle ranging from ϑ to $(\vartheta + d\vartheta)$, over the whole sky. The factor of $\frac{1}{2}$ removes the "double-counting" redundancy caused by the fact that the pairs "i-j" and "j-i" are essentially the same. As a sanity check, integrating Eq. (7) over the whole sky gives us the well known expression for the number of two objects, C_N, picked from a selection of N objects (here stars):

$$C_N = \int_0^\pi df_{ij}(\vartheta) = \frac{N(N-1)}{2} \tag{8}$$

We are now in the position to obtain an expression for the frequency, $f\,(\vartheta_{ij}, K\sigma)$, of star-pairs lying in the error band, $[(\vartheta_{ij} - K\sigma), (\vartheta_{ij} + K\sigma)]$ by integrating Eq. (7) over the small region defined by the error band:

$$f\,(\vartheta_{ij}, K\sigma) = \frac{N(N-1)}{4} \int_{\vartheta_{ij}-K\sigma}^{\vartheta_{ij}+K\sigma} \sin\vartheta\, d\vartheta = \frac{N(N-1)}{2}\, \sin K\sigma\, \sin\vartheta_{ij} \tag{9}$$

This equation represents *the expected frequency of star-pairs in the catalog (with the specified inter-star angle ϑ_{ij}) that may be matched incorrectly to the observed star-pair, to within the measurement precision specified by the band $K\sigma$.*

We digress here a little to note that the actual star distribution is not uniform in practice. However, study of the actual star position density indicates that perhaps a factor of two or more star density frequency variations from a whole sky uniform density model occur, when averaged over the typical size of a star camera field of view. Thus we can take these variations into account by conservatively interpreting the frequency results obtained (i.e., in practice we may be able to tolerate the difference between sufficiently small frequencies such as 10^{-10} or 10^{-12}, by using the star catalog to bound worst case errors and using the formulas derived).

In what follows, we develop generalizations of Eq. (9) so that the objects of interest are general star polygons containing $n > 2$ stars.

Figure 3 shows, for StarNav I experiment[‡], the residuals between the values for f_{ij} provided by Eq. (9) and simulated data from random star catalog access with various star separation angles. After some experimentation, trading computational expense versus our desire to essentially never fail to access the measured stars in the star catalog, we found that a K-value of about $K = 4.5\sqrt{2}$ is a good compromise. This value which is conservatively 50% greater than the perhaps most predictable 3-σ choice we might make of $k = 3\sqrt{2}$ (which can be derived for an inter-star angle associated with two stars whose direction measurement errors are both normally distributed with the same centroiding error variance of σ^2). The

[‡]8 deg square FOV, magnitude threshold $m = 5.5$, 512×512 pixel focal plane detector, focal length = 50 mm, and $3\sigma = 10$ arcsec. This camera and star identification experiment flew successfully aboard the ill-fated Columbia STS-107.

4.5-σ K-value has been adopted to almost certainly guarantee that the actual star pair measured is contained in the k-vector star subset with an expected number of elements = f_{ij}. Adopting such a highly conservative K value results in a corresponding increase in the length of the k-vector candidate star subsets and an associated increase in the computational burden to identify stars, but ensures that we obtain essentially of all the possible measured stars as candidate stars, even with the actual non-uniform star position density. For the exceptionally rare circumstance of very large statistically outlying star measurement errors (> 4.5-σ), when even such a conservative catalog access does not contain all of the measured star pairs, we can be comforted by the practical truth that most advanced star trackers measure many redundant stars in each image, so such highly infrequent failures to access and identify every measured star will result in negligible reduction in performance.

For nominal parameter settings (see below), we find $f_{ij} \cong 200$ is typical, so we see that $\sigma_{f_{ij}} \cong 0.1 f_{ij}$ and apparently the approximation of uniform star position distribution density on the celestial sphere does lead to moderate errors. However, our studies indicate the uniform density approximation is more than adequate for order of magnitude expected frequency analysis. We note that a future extension of this work can address the known departure of the actual star position density distribution from the uniform distribution assumed herein, which ultimately will lead to increased accuracy in the further generalized analogies of the frequency formulas derived herein.

EXPECTED CATALOG MATCH FREQUENCY FOR A STAR PATTERN WITH TWO LEGS

Consider the case of a three star pattern ijk. We seek to match the measured inter-star angles $(\vartheta_{ij} \pm K\sigma)$ and $(\vartheta_{ik} \pm K\sigma)$. To do this, let us consider "pivoting" about the i-th measured star. Thus, using Eq. (6), it is easy to evaluate the number of stars, $\bar{f}_{ij}$, displaced from i by an angle varying between $(\vartheta_{ij} - K\sigma)$ to $(\vartheta_{ij} + K\sigma)$:

$$\bar{f}_{ij} = \int_{\vartheta_{ij}-K\sigma}^{\vartheta_{ij}+K\sigma} dn(\vartheta) = (N-1)\,\sin K\sigma\,\sin\vartheta_{ij} \tag{10}$$

Note that the above equation is different from Eq. (9) in the sense that in Eq. (10), the star "i" is *fixed*, whereas in Eq. (9), all possible candidates for the i-th star have been considered. In the same manner, the expected number $\bar{f}_{ik}$ of stars displaced from i by an angle which varies from $(\vartheta_{ik} - K\sigma)$ to $(\vartheta_{ik} + K\sigma)$ is

$$\bar{f}_{ik} = \int_{\vartheta_{ik}-K\sigma}^{\vartheta_{ik}+K\sigma} dn(\vartheta) = (N-1)\,\sin K\sigma\,\sin\vartheta_{ik} \tag{11}$$

Now we consider all possible candidates for the i-th star, which can be any one of the N cataloged stars. Doing so, the expected frequency that a measured star matches two neighboring stars with both legs (star pairs ij and ik) is:

$$f_{i-(j,k)} = N\,\bar{f}_{ij}\,\bar{f}_{ik} = N\,[(N-1)\,\sin K\sigma]^2\,\sin\vartheta_{ij}\,\sin\vartheta_{ik} \tag{12}$$

Some comments are in place here. Obviously, note that $\vartheta_{ij} \neq \vartheta_{ik}$, or else the two stars j and k lie in the same $K\sigma$ strip for star i. Specifically, the validity condition for Eq. (12)

is $|\vartheta_{ij} - \vartheta_{ik}| > 2K\sigma$. Also, notice that Eq. (12) does not provide the frequency for invalid match of a star triangle. Instead, it provides the frequency of matching only two legs with one end at star i.

EXPECTED CATALOG MATCH FREQUENCY FOR STAR PATTERN WITH M LEGS

Equation (12) can be easily generalized to the expected frequency that a star matches with M legs (that is, with M other stars identified by), giving us:

$$f_{i-(j_1,\ldots,j_M)} = N \prod_{j=j_1}^{j_M} \bar{f}_{ij} = N\left[(N-1)\,\sin(K\sigma)\right]^M \prod_{j=j_1}^{j_M} \sin\vartheta_{ij} \tag{13}$$

that complete the searched solution.

We reiterate that the above results, Eqs. (12) and (13), do not match all the inter-star angles in the closed star polygon. Therefore these formulas may admit many more expected spurious match events than necessary, because they have not used all of the measured information and are thus conservative. The frequency count for closed star patterns is addressed below, beginning with the most fundamental case of a three star measured pattern.

EXPECTED CATALOG MATCH FREQUENCY FOR A STAR TRIANGLE

When trying to identify a star triangle, there is a possibility of the existence of a mirror image triangle. This possibility can be eliminated by checking to see if the cataloged triple (r_1, r_2, r_3) is consistent with the associated measured triple (b_1, b_2, b_3). The two triangles are then consistent if the following condition

$$\text{sign}[r_1^{\text{T}}(r_2 \times r_3)] = \text{sign}[b_1^{\text{T}}(b_2 \times b_3)] \tag{14}$$

is satisfied. Upon checking Eq. (14), only the triple star pattern identifications which passes this test are further considered.

Let us now consider the frequency of random occurrence associated with matching all three angles of a star triangle. With reference to Figure 4, let us consider one of the f_{ij} star pairs, where f_{ij} has the expression provided by Eq. (9), and let us consider the intersection area δA of the two spherical surfaces associated with the angles ϑ_{ik} (centered at the i-star) and ϑ_{jk} (centered at the j-star). Notice for the given measured angles, and associated uncertainties, the k-th star must lie in one of the small areas δA. In order to get an estimate of the area δA, consider Figure 5. The area δA is the one enclosed by the bold solid lines on the surface of the celestial sphere. We can write an approximate expression for this area as follows:

$$\delta A \approx (\theta_i^+ + \theta_i^-)(\theta_j^+ + \theta_j^-) \sin\theta_{ik} \sin\theta_{jk} \tag{15}$$

The angles corresponding to star i have been shown in Figure 5. Angles for star j are defined analogously. Using spherical trigonometry, we have the following (exact) expressions for the various angles appearing in Eq. (15):

$$\theta_{ik}^{+} = \cos^{-1}(\cos K\sigma \cos\theta_{ik} + \sin K\sigma \sin\theta_{ik} \cos\theta_{k}) \tag{16}$$

$$\theta_{ik}^{-} = \cos^{-1}(\cos K\sigma \cos\theta_{ik} - \sin K\sigma \sin\theta_{ik} \cos\theta_{k}) \tag{17}$$

$$\theta_{i}^{+} = \sin^{-1}\left(\frac{\sin K\sigma \sin\theta_{k}}{\sin\theta_{ik}^{+}}\right) \tag{18}$$

$$\theta_{i}^{-} = \sin^{-1}\left(\frac{\sin K\sigma \sin\theta_{k}}{\sin\theta_{ik}^{-}}\right) \tag{19}$$

Similar expressions can be obtained for angles θ_j^+ and θ_j^-. Thus, the expected frequency of random occurrence that a given star triangle is matched within measurement precision is given by:

$$f_{ij-k} = f_{ij}\rho^{**}\delta A \tag{20}$$

$$= \frac{N(N-1)}{2} \sin K\sigma \sin\theta_{ij} \frac{(N-2)}{4\pi}\delta A \tag{21}$$

where $\rho^{**} = (N-2)/(4\pi)$ is a modified star density that *does not take into account* the stars i and j that obviously cannot fall into the small δA cone intersection area. Using Eq. (15) to approximate δA, we have that

$$f_{ij-k} = \frac{N(N-1)(N-2)}{8\pi}(\theta_i^+ + \theta_i^-)(\theta_j^+ + \theta_j^-)\sin K\sigma \sin\theta_{ij} \sin\theta_{ik} \sin\theta_{jk} \tag{22}$$

We note again that Eq. (15) is an approximate estimate of the concerned area of intersection in which the k-th star lies. It is possible to obtain an exact expression in terms of differences of spherical lenses as shown in Figure 6. Clearly, the area of concern, δA is given by the following expression:

$$\delta A_{exact} = \frac{1}{2}\left[\,(L1 - L2) - (L3 - L4)\,\right] \tag{23}$$

STAR POLYGONS EXCEEDING THREE STARS

For star patterns exceeding three stars, the analytical expressions are somewhat complicated. The basic idea for any M star ($M > 2$) structure is to obtain the frequency of finding incorrect star matches for the $(M-1)$ star structure, and then to multiply it with the frequency of obtaining an incorrect match for the M^{th} star relative to the $(M-1)$ star structure. Notice that the area on the celestial sphere in which the last (M^{th}) star can be found gets smaller and smaller as M increases and it becomes increasingly difficult to obtain its exact value. Without introducing any simplifying assumptions, the area in which the M^{th} star lies is the intersection of the $[-K\sigma, K\sigma]$ error bands with each of the other $(M-1)$ stars as pivots. A simple estimate of this area can be obtained by taking the minimum of all possible area intersections taking two stars at a time. This allows us to utilize the formula obtained in Eq. (15) and use the minimum over all star-pairs. Notice that this is a conservative estimate, since $\bigcap_i \delta A_i \leq \min_i \delta A_i$. Mathematically, we have:

$$f_{i_1 i_2 \ldots i_{M-1}-(i_M)} = f_{i_1 i_2 \ldots i_{M-2}-(i_{M-1})} \rho_{M-1} \delta A_M \tag{24}$$

where ρ_{M-1} is the star density obtained upon excluding the first $M-1$ stars, which are known and no longer contribute to uncertainty. Consequently, we have $\rho_{M-1} = [N - (M-1)]/4\pi$. As described above, δA_M is the area in which the M^{th} star resides, given the $(M-1)$ star structure. The "two-star" estimate of this area is given by:

$$\delta A_M = \min_i \{\delta A(i_1, i_2), \delta A(i_1, i_3), \ldots, \delta A(i_1, i_{M-1}), \ldots, \delta A(i_{M-2}, i_{M-1})\} \tag{25}$$

where, each of $\delta A(m,n)$ is computed using Eq. (15). In Eq. (24), there is some subjectivity since we can form the $(M-1)$ star polygon by leaving out any one of the M stars, leading to identical formulae based on different inter-star angle combinations. However, since we use these equations for order of magnitude analysis only, any of these frequency formulas is appropriate. Also, to avoid consideration of reflections, each cataloged and measured triple from the hypothesized match of p measured and cataloged stars should be tested using Eq. (14), all hypothesized identification of triples that do not pass these tests should be rejected.

EXAMPLE: FOUR - STAR POLYGON

Let the stars be indexed by the letters i, j, k and l. The number of possible incorrectly matched triangles is given by $f_{ij-(k)}$ found in the previous section. Clearly then, using Eq. (24), we get the following expression for the frequency of obtaining incorrect matches for the 4-star polygon:

$$f_{ijk-(l)} = f_{ij-(k)} \rho_3 \min\{\delta A(i,j), \delta A(i,k), \delta A(j,k)\} \tag{26}$$

In Eqs.(9, 22, 26), we see the inherent competition between the size of the catalog (N), and the precision of the measurement ($K\sigma$) - as a larger N tends to increase the frequencies and smaller ($K\sigma$) tends to decrease the frequencies. Looking at the dependency of the frequency relationships on the error band $K\sigma$, it is evident that we can drive the frequency of incorrect random matches arbitrarily small if we are able to consider a large number of measured stars.

Expected catalog match frequency for the general star structure

For the most general case, the following two formulas:

- Equation (13), which gives the closed form expression of the frequency of a star with M legs, and
- Equation (24), which provides the expression of a M-star star polygon (of which the simplest case is the triangle),

allow us to build other more complicated cases. Let us give by means of an example, the general idea on how to proceed. Consider seven measured stars identified by the indices i, j, k, l, m, n, and r, and let us quantify the frequency associated with only a subset of all the possible considered legs. One possibility could be that we consider all the angular legs of the triangle (i-j-k), and all the angular legs between the stars (l-m-n-r) with the triangle (i-j-k), but *not* the angles among the stars (l-m-n-r). Thus we have a three star polygon (i.e. triangle) and four three-leg star patterns centered at l, m, n and r respectively. The frequency for this selection is simply obtained by the simultaneous application of the above mentioned formulae. In this case we will have:

$$f = f_{i,j-k}\ f_{l-(i,j,k)}\ f_{m-(i,j,k)}\ f_{n-(i,j,k)}\ f_{r-(i,j,k)} \tag{27}$$

Notice that the above described case is an illustration of how much data we wish to consider in obtaining the error probability. If the information left out is significant, the estimate will be extremely conservative. The point however is that we would have the flexibility to consider any star structure constructed out of a number of measured stars. Generally speaking, we should base accepting a hypothesis for star identification as valid (based on a given pattern of measured inter-star angles matching those from a cataloged pattern to within measurement error), when the frequency of a random match between the measured and cataloged patterns is less than some tolerance. Note we may frequently satisfy this criterion with fewer than the maximum number of inter-star angles among the observed stars.

NUMERICAL EXAMPLE

To see the implications of these developments, consider the following typical values: $N = 5,000, \sigma = 25$ μrad, $k = 6.4$. Let us consider 5 stars indexed by i, j, k, l and m. We consider nominal values for the various measured inter-star angles, i.e. $\vartheta_{ij}, \vartheta_{ik}, \vartheta_{il}, \vartheta_{im}, \vartheta_{jk}$ etc. Various star polygon results are shown in Table 1 with comparisons to Monte Carlo results. Table shows results for the open star structure with 2, 3, and 4, legs along and a comparison to the Monte Carlo method.

We see that the frequencies match well with Monte Carlo analysis (when feasible). We mention that it is not feasible to obtain high-confidence results for polygons with more than 3 stars with the Monte Carlo approach because of the large number of sample points required to obtain a converged result. It is clear from Table 1 that matching of four or five star patterns results in a very low expected frequency of occurrence that an incorrect random pattern from the star catalog could have matched the measured star pattern. Said another way, matching a measured star pattern with four or more stars to within measurement precision, we approach

Table 1: Numerical results for star polygons with 2, 3, 4, and 5 Stars. $N = 5,000$ stars in catalog.

σ	Method	f_{ij}	f_{ij-k}	f_{ijk-l}	f_{ikjl-m}
25 μrad	Analytical	845	2.2×10^{-2}	3.9×10^{-7}	9.2×10^{-12}
25 μrad	Monte Carlo	835	Not Feasible	Not Feasible	Not Feasible
250 μrad	Analytical	8450	22	3.8×10^{-2}	9.2×10^{-5}
250 μrad	Monte Carlo	8391	33	Not Feasible	Not Feasible

engineering certainty that we have the correct star identification, especially if star patterns in successive frames have such matches and there is some overlap in the pattern matched star-fields. We reiterate that the above analytical results for star polygons exceeding three stars are considerably conservative because of the "minimum"-approximation used in Eq. (25) to compute the area of intersection in which every additional star introduced lies. To get of feel of such conservatism, we increased the error band to 2000 μrad and reduced the number of stars in the catalog to 2000 stars to perform a rough Monte Carlo counting analysis. We found that the expressions provided in this paper led to 32 incorrect matches for the 4-star polygon while the Monte Carlo analysis resulted in only 2 false matches. Note that the error band (σ) will not assume such a high value in a real-life situation.

Also notice from Table that the open star leg structures have high probability of incorrect matches because all inter-star angles are not considered. The Monte Carlo results match extremely well in this case because there are no simplifying assumptions used in deriving

Table 2: Numerical results for the star-leg structure pattern match with 2, 3, and 4 legs, and star "i" at the vertex. $N = 5,000$ stars in catalog and $\sigma = 25\,\mu$rad.

Method	$f_{i-(j,k)}$	$f_{i-(j,k,l)}$	$f_{i-(j,k,l,m)}$
Analytical	304	67	11
Monte Carlo	308	69	11

the concerned formulae. Comparing Table to the first row of Table 1, it is evident that including all inter-star angles in the star identification pattern match is vital to obtain high confidence identification with only 4 or 5 stars.

CONCLUSIONS

In this paper, we have introduced a novel method to quantify the frequency of false matching in star pattern identification, based on the assumption of uniform star distribution over the celestial sphere. We have introduced an analytical means to compute *the expected frequency of random occurrence* that a cataloged polygon of stars could possibly match, to within camera precision, the given measured polygon. This analytical means of computing the expected frequency is novel and important to eliminate the need for expensive and slowly converging Monte Carlo estimates of star identification reliability.

In particular, we derive the expected random frequencies associated with matching inter-star angles from measured star polyhedra. Using recursively the frequency analysis for matching triangles, we show how to develop the expected frequency for four and five star patterns with associated formulas derived for the expected frequency of random matches to within measurement precision. These formulas are original contributions which permit, for the first time, a rigorous analytical basis for deciding upon the validity of an identified star pattern with knowledge of the frequency that an invalid match could occur.

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